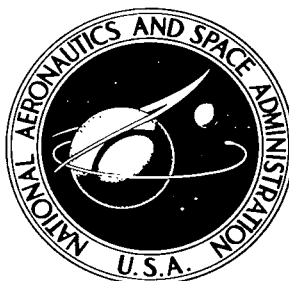


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A SERIES SOLUTION FOR SOME
PERIODIC ORBITS IN THE RESTRICTED
THREE-BODY PROBLEM ACCORDING
TO THE PERTURBATION METHOD

by Su-Shu Huang

Goddard Space Flight Center

Greenbelt, Md.

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

A series is obtained for those periodic orbits surrounding the more massive of the two finite bodies in the restricted three-body problem. The expansion is in terms of the mass of the less massive finite body. The initial conditions predicted by the series for several periodic orbits are compared with those obtained by purely numerical processes. They are in good agreement for the case corresponding to the earth-moon system. Also, a simple theory of nearly periodic orbits in the neighborhood of a periodic orbit is developed and numerically verified by examples. Finally, it is suggested that, if asteroids avoid places where their orbits would be commensurable with the period of Jupiter, then artificial satellites and dust particles may avoid certain areas around the earth as a result of the presence of the moon.

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INTRODUCTION

A periodic solution may be regarded as a solution of the differential equations of motion that satisfies, in addition to the initial conditions, the condition that after a lapse of one period, P , both coordinates and velocities return to their initial values. Thus, the problem of finding a periodic orbit in celestial mechanics resembles the problem of finding the eigen function for an eigen value in quantum mechanics, since the eigen value is determined by boundary conditions. Indeed, it is this basic concept that led to the derivation of a series solution for those periodic orbits in the restricted three-body problem that are revolving around the more massive of the two finite mass points. The recent papers by Szebehely give a detailed discussion of the restricted three-body problem (References 1 and 2).

The mathematical method used here follows the standard technique employed in classical mechanics under these circumstances. Indeed, it is very similar to the method Hill used in his lunar theory (References 3 and 4; also see Reference 5). Actually, the present analysis is somewhat parallel to the analysis that led Hill to his variation orbit, for both his and the present method depend upon some series expansions. However, in his lunar theory Hill expanded the solution in terms of the ratio of the mean motion of the sun to that of the moon in the rotating coordinate system. In this present theory for artificial satellites orbiting the earth in the earth-moon system the solution is expanded in terms of the mass of the moon. The mathematical simplicity in terms of the lunar mass is obvious; however, the result cannot be applied to the study of the motion of the moon because the moon is revolving around a relatively much less massive body (i.e., the earth) in the earth-sun system; consequently, it is senseless to expand the solution in terms of the solar mass.

EQUATIONS OF MOTION

The total mass of the two finite bodies is considered the unit of mass, and their separation the unit of length. The unit of time is such that the gravitational constant is unity. If we adopt

the line joining the two finite mass points as the x-axis and the location of the greater of these masses as the origin, the equations of motion of the third, infinitesimal body in this rotating system of reference become:

$$\frac{d^2 x}{dt^2} - 2 \frac{dy}{dt} = x - \mu - (1-\mu) \frac{x}{(r_1')^3} - \mu \frac{x-1}{(r_2')^3}, \quad (1)$$

$$\frac{d^2 y}{dt^2} + 2 \frac{dx}{dt} = y - (1-\mu) \frac{y}{(r_1')^3} - \mu \frac{y}{(r_2')^3}, \quad (2)$$

where it is assumed that the mass $1-\mu$ is at the origin. Consequently the mass μ is at point $(1, 0)$. Also, r_1' and r_2' are the distances of the third body from the $1-\mu$ and μ components, respectively. It is obvious that r_1' is also the distance of the third body from the origin; so

$$r_1' = r. \quad (3)$$

We are interested in the periodic orbits revolving around the $1-\mu$ component. Therefore, we make the transformations

$$x = r \cos \theta, \quad (4)$$

$$y = r \sin \theta. \quad (5)$$

This brings Equations 1 and 2 into the forms

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 - 2r \frac{d\theta}{dt} = r - (1-\mu) \frac{1}{r^2} - \mu \frac{r}{(r_2')^3} + \mu \left[\frac{1}{(r_2')^3} - 1 \right] \cos \theta, \quad (6)$$

$$r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} + 2 \frac{dr}{dt} = -\mu \left[\frac{1}{(r_2')^3} - 1 \right] \sin \theta. \quad (7)$$

EQUATIONS OF PERTURBATION

When μ equals zero all circles with their center at the origin are the periodic solutions of the problem. We can now argue that if μ is small, the periodic orbit deviates only slightly from a circular one. The deviation obviously depends upon μ . Therefore, the periodic solutions may be written as

$$r = r_0 + \mu r_1(t) + \mu^2 r_2(t) + \dots, \quad (8)$$

$$\theta = \lambda t + \mu \theta_1(t) + \mu^2 \theta_2(t) + \dots, \quad (9)$$

where r_0 and λ are to be determined and are independent of time. Substituting Equations 8 and 9 into Equations 6 and 7 gives equations of different orders of approximation after a series of long but straightforward calculations.

The zeroth order (μ^0) approximation gives

$$(\lambda + 1)^2 = \frac{1 - \mu}{r_0^3} , \quad (10)$$

which is simply Kepler's third law in the problem of two bodies, the $1 - \mu$ component and the third, infinitesimal body. The term is $\lambda + 1$ instead of λ because the equations are expressed in the rotating coordinate system. Also, λ may have positive or negative values, corresponding respectively to the direct and retrograde motion of the third body.

The first order (μ^1) approximation yields

$$\frac{d^2 r_1}{dt^2} - 2(\lambda + 1) r_0 \frac{d\theta_1}{dt} - 3(\lambda + 1)^2 r_1 = \frac{1}{2} r_0 + \frac{3}{2} r_0 \cos 2\lambda t + \frac{1}{8} r_0^2 (9 \cos \lambda t + 15 \cos 3\lambda t) , \quad (11)$$

$$r_0 \frac{d^2 \theta_1}{dt^2} + 2(\lambda + 1) \frac{dr_1}{dt} = -\frac{3}{2} r_0 \sin 2\lambda t - \frac{3}{8} r_0^2 (\sin \lambda t + 5 \sin 3\lambda t) , \quad (12)$$

if the terms involving third and higher orders of r_0 are neglected.

With the same degree of approximation in regard to the series in r_0 , the second order (μ^2) equations are:

$$\begin{aligned} \frac{d^2 r_2}{dt^2} - 2(\lambda + 1) r_0 \frac{d\theta_2}{dt} - 3(\lambda + 1)^2 r_2 &= r_0 \left(\frac{d\theta_1}{dt} \right)^2 + 2(\lambda + 1) r_1 \frac{d\theta_1}{dt} - 3(\lambda + 1)^2 \frac{r_1^2}{r_0} \\ &+ \frac{1}{2} r_1 (1 + 3 \cos 2\lambda t) + \frac{1}{4} r_0 r_1 (9 \cos \lambda t + 15 \cos 3\lambda t) \\ &- 3r_0 \theta_1 \sin 2\lambda t - \frac{r_0^2 \theta_1}{8} (45 \sin 3\lambda t + 9 \sin \lambda t) , \end{aligned} \quad (13)$$

and

$$\begin{aligned} r_0 \frac{d^2 \theta_2}{dt^2} + 2(\lambda + 1) \frac{dr_2}{dt} &= -2 \frac{dr_1}{dt} \frac{d\theta_1}{dt} - r_1 \frac{d^2 \theta_1}{dt^2} - \frac{3}{2} r_1 \sin 2\lambda t - 3r_0 \theta_1 \cos 2\lambda t \\ &- \frac{1}{4} r_0 r_1 (3 \sin \lambda t + 15 \sin 3\lambda t) - \frac{1}{8} r_0^2 \theta_1 (3 \cos \lambda t + 45 \cos 3\lambda t) . \end{aligned} \quad (14)$$

In a similar way equations of higher orders can be derived in terms of the solutions of the equations of lower orders.

THE FIRST ORDER EQUATIONS

The solutions of Equations 11 and 12 can be easily found:

$$r_1 = -\frac{r_0}{6(\lambda+1)^2} + A_1' \cos(\lambda+1)t + A_2' \sin(\lambda+1)t - \frac{2B_1'}{3(\lambda+1)} + k_1 \cos \lambda t + k_2 \cos 2\lambda t + k_3 \cos 3\lambda t, \quad (15)$$

$$r_0 \theta_1 = l_1 \sin \lambda t + l_2 \sin 2\lambda t + l_3 \sin 3\lambda t + B_1' t + B_2' - 2A_1' \sin(\lambda+1)t + 2A_2' \cos(\lambda+1)t. \quad (16)$$

A_1' , A_2' , B_1' , and B_2' are arbitrary constants which make Equations 15 and 16 the most general solution; k_n and l_n are defined by

$$k_1 = \frac{15+6}{8\lambda(2\lambda+1)} r_0^2, \quad (17)$$

$$k_2 = -\frac{3(2\lambda+1)}{2\lambda(3\lambda^2-2\lambda-1)} r_0, \quad (18)$$

$$k_3 = -\frac{5(5\lambda+2)}{8\lambda(8\lambda^2-2\lambda-1)} r_0^2, \quad (19)$$

$$l_1 = -\frac{3(10\lambda^2+12\lambda+3)}{8\lambda^2(2\lambda+1)} r_0^2, \quad (20)$$

$$l_2 = \frac{3(11\lambda^2+10\lambda+3)}{8\lambda^2(3\lambda^2-2\lambda-1)} r_0, \quad (21)$$

$$l_3 = \frac{5(6\lambda^2+4\lambda+1)}{8\lambda^2(8\lambda^2-2\lambda-1)} r_0^2. \quad (22)$$

It should be observed that λ (or, equivalently, r_0), which appears in Equation 10, is not an integration constant in the perturbed case, although it is in the unperturbed case. Indeed, in the unperturbed case λ is the only integration constant that does not vanish under the assumed initial conditions. The integration constants in the perturbed case are A_1' , A_2' , B_1' , and B_2' . In general, if higher orders of μ are considered, A_1 , A_2 , B_1 , and B_2 are defined by

$$A_1 = A_1' \mu + A_1'' \mu^2 + \dots, \quad \text{etc.},$$

because, as we shall see later, the complementary functions in different orders of approximation are of the same form. Consequently, the integration constants in different orders of μ can be

combined into four arbitrary constants A_1 , A_2 , B_1 , and B_2 to agree with the expected number of four that occur in the solution of two second order differential equations (Equations 6 and 7).

The integration constants A_1 , A_2 , and B_2 are not particularly relevant, but it is important to note the dynamical meaning of B_1 . If λ' is the mean angular velocity in the perturbed case, we have, from Equations 9 and 16,

$$\lambda' = \lambda + \frac{B_1' \mu}{r_0}$$

or in general

$$\lambda' = \lambda + \frac{B_1}{r_0} .$$

Therefore, $B_1' \mu / r_0$ (or B_1 / r_0) is the change in the mean angular velocity from the unperturbed case to the perturbed case. Hence the four integration constants for the perturbed case may be taken as A_1 , A_2 , λ' , and B_2 (or A_1' , A_2' , λ' , and B_2' in the first approximation), and λ , or equivalently r_0 , acts in the perturbed case as a standard for comparison with the perturbed orbits and is consequently a parameter.

We can set $\lambda' = \lambda$ for the sake of simplicity because both the perturbed and unperturbed periodic orbits form a continuous family. This leaves $B_1' = 0$.

Even with $B_1' = 0$, the general solution does not give the periodic orbits because there are two fundamental periods $2\pi/(\lambda + 1)$ and $2\pi/\lambda$, with several harmonics of the latter. However, since the existence of some periodic solutions has been proved (see for example Reference 1), these periodic orbits must correspond to the particular integral obtained from Equations 15 and 16 by setting $A_1' = A_2' = B_1' = B_2' = 0$. Then the solutions contain only terms with period $2\pi/\lambda$ of the fundamental oscillation, and shorter periods corresponding to its harmonics. Thus the periodic orbits around the $1 - \mu$ component may be given, to the first order of μ and the second order of r_0 , by

$$r = r_0 + \mu r_1 , \quad (23)$$

$$\theta = \lambda t + \mu \theta_1 , \quad (24)$$

where

$$r_1 = - \frac{r_0}{6(\lambda + 1)^2} + k_1 \cos \lambda t + k_2 \cos 2\lambda t + k_3 \cos 3\lambda t , \quad (25)$$

$$r_0 \theta_1 = l_1 \sin \lambda t + l_2 \sin 2\lambda t + l_3 \sin 3\lambda t . \quad (26)$$

It follows from the solution given by Equations 15 and 16 that, in general, a periodic solution can be obtained for any given value of λ only by setting A_1 , A_2 , B_1 , and B_2 equal to zero. Thus, one periodic orbit is associated with one value of the period. However, if λ is a ratio of two integers, periodic orbits may exist for arbitrary (small) values of these constants. If they do, a large number of periodic orbits would be found for some particular values of the period.

THE SECOND ORDER EQUATIONS

When the solutions of r_1 and $r_0 \theta_1$ given by Equations 25 and 26 are substituted into the second order equations (13 and 14) and the resulting equations are simplified, we obtain:

$$\frac{d^2 r_2}{dt^2} - 2(\lambda + 1) r_0 \frac{d\theta_2}{dt} - 3(\lambda + 1)^2 r_2 = \beta_0 + \sum_{n=1} \beta_n \cos n\lambda t, \quad (27)$$

$$r_0 \frac{d^2 \theta_2}{dt^2} + 2(\lambda + 1) \frac{dr_2}{dt} = \sum_{n=1} \rho_n \sin n\lambda t. \quad (28)$$

These equations have the same form as Equations 11 and 12, except for more terms on the right-hand side. Therefore, the solution can be derived in the same manner as in the case of the first order equations, although finding the explicit expressions of the solutions is much more tedious because of the lengthy equations that define β_n and ρ_n . The author has evaluated only β_0 . From this the average radius of the periodic orbit may be derived:

$$\langle r \rangle = r_0 \left\{ 1 - \frac{\mu}{6(\lambda + 1)^2} + \mu^2 \left[\frac{1}{18(\lambda + 1)^4} + \frac{3(19\lambda^2 + 14\lambda + 3)}{32\lambda^2 (\lambda + 1)^2 (3\lambda^2 - 2\lambda - 1)} + \frac{9(2\lambda + 1)^2}{8\lambda^2 (3\lambda^2 - 2\lambda - 1)^2} \right] \right\}, \quad (29)$$

correct to the second order both in μ and r_0 .

It follows from Equation 29 that for a given value of λ the average radius of the periodic orbit around the earth in the presence of the moon is slightly less than that given by Equation 10, which is for the case of the absence of the moon. This fact may be easily understood because the overall long-range effect of the moon on a satellite that is revolving around the earth is to reduce the central attractive force of the earth on the satellite. The argument becomes physically apparent if we imagine the moon and its orbit to be replaced by an annular ring of the same mass as the moon, with this mass uniformly distributed. From this reasoning the presence of the sun may be expected to further reduce, by a small amount, the average value of the radius of the satellite's orbit. The prediction has been verified by actual calculations.

The similarity between the present calculation and the perturbation theory in quantum mechanics is apparent. In neither case is the convergence of the series solution proven. But, it

will be shown in the following sections that the periodic orbits derived in this way agree perfectly with those obtained by the trial and error method, just as the effectiveness of the perturbation method in quantum mechanics is based on its ability to predict empirical results.

On the other hand, the present perturbation method differs in many ways from that in quantum mechanics. For example, the main purpose of the perturbation theory in quantum mechanics is to find the new eigen value as a result of perturbation, whereas here we are interested in the variation in the orbital nature for a given value of λ .

Although the approach parallels Hill's determination of the variation orbit, differences do exist. In the first place, Hill started with a set of differential equations already approximated by the neglecting of terms involving the ratio of the mean distance of the moon to that of the sun. The present investigation uses the equations in the restricted three-body problem. Secondly, Hill was concerned only in obtaining a particular solution but we are interested in the general solution that involves four arbitrary constants, A_1 , A_2 , B_1 , and B_2 , which determine, as shall be seen later, those nearly periodic orbits in the neighborhood of the exact periodic solution. Consequently, the present solution gives an entire family of orbits in the neighborhood of any periodic solution that we can determine. Needless to say, for orbits that depart greatly from the periodic one, the approximation employed in this analysis breaks down. Consequently, those orbits can no longer be represented by the equations derived.

The differences between Hill's analysis and the present one clearly reflect divergent problems faced in different times. Indeed, the present solution of periodic and nearly periodic orbits around a more massive component, expanded in terms of the mass of the less massive component, would have had little practical significance in Hill's time.

NUMERICAL APPLICATIONS

In a previous paper (Reference 6), by a numerical method (Reference 7) a synchronous orbit around the earth was derived, under the idealization of the restricted three-body problem with $\mu = 0.012149$. Now we are able to derive it from Equations 23-26.

In the XY coordinate system with the origin at the mass point $1 - \mu$, the initial conditions of the synchronous orbit, derived by successive approximations, are

$$\left. \begin{aligned} x_0 &= 0.10959080 , \\ y_0 &= 0 , \\ \dot{x}_0 &= 0 , \\ \dot{y}_0 &= 2.8927303 , \end{aligned} \right\} \quad (30)$$

Table 1

Values of k_n and l_n for
 $\mu = 0.012149$ and $\lambda = 26.396884$.

n	k_n	l_n
1	4.249394×10^{-4}	8.756117×10^{-4}
2	-1.644862×10^{-4}	2.296898×10^{-4}
3	-6.901187×10^{-6}	8.365787×10^{-6}

which yield

$$P = 0.23802754 \quad (31)$$

for the period.

We can now calculate for this case the values of k_n and l_n from Equations 17-22, with

$$\lambda = \frac{2\pi}{P} = 26.396884, \quad (32)$$

which gives $r_0 = 0.10958800$ from Equation 10.

The computed values are listed in Table 1. Sub-

stituting these values of k_n and l_n in the solution given by Equations 23-26 and transforming variables from r and θ to x and y , in accordance with Equations 4 and 5, shows that the values $x(t), y(t), \dot{x}(t)$, and $\dot{y}(t)$ derived from the present formulas agree to seven significant figures with those obtained from direct integration under the initial conditions given by Equations 30. In particular, at $t = 0$ our formulas predict the initial conditions

$$\left. \begin{aligned} x_0 &= 0.10959080, \\ y_0 &= 0, \\ \dot{x}_0 &= 0, \\ \dot{y}_0 &= 2.8927300, \end{aligned} \right\} \quad (33)$$

for the periodic orbit with P given by Equation 31. The agreement between Equations 30 and 33 must be regarded as satisfactory.

As would be expected, the prediction of periodic orbits by Equations 23 and 24 becomes less and less accurate as the period increases. A few cases are given in Table 2 to show the

Table 2

Initial Conditions Derived by the Series Solution Compared with those Obtained by the Method of Successive Approximation ($\mu = 0.012149, y_0 = \dot{x}_0 = 0$).

P	Series Solution		Successive Approximation		
	x_0	\dot{y}_0	x_0	\dot{y}_0	Jacobian Constant
0.23802754	0.10959080	2.8927300	0.10959080	2.8927303	9.6968861
0.39999890	0.15239388	2.3936373	0.15239410	2.3936354	7.2832706
0.59999666	0.19580805	2.0503747	0.19580870	2.0503734	5.9498741
0.79999290	0.23271671	1.8277779	0.23271790	1.8277824	5.2292161
0.99999026	0.26506832	1.6657674	0.26506980	1.6657871	4.7756998
1.1999940	0.29394940	1.5398118	0.29395020	1.5398618	4.4638648
1.4000154	0.32005211	1.4376335	0.32005020	1.4377350	4.2365564
1.6000716	0.34385494	1.3522620	0.34384680	1.3524439	4.0638285

progressive worsening of the prediction. However, it should be noted that even at $P = 1.6$, the two sets of calculations — one from the numerical approach and the other from the present formulas — still give results that agree to the fourth significant figure.

ORBITS IN THE NEIGHBORHOODS OF THE PERIODIC ONES

It is obvious that for orbits which are not exactly periodic, the general solution given by Equations 23 and 24 together with Equations 15 and 16 should be applied instead of the particular integral given by Equations 23-26. Let us now consider the behavior of these orbits when we make the initial values of x_0 and \dot{y}_0 only slightly different from those corresponding to the periodic orbit, while maintaining $y_0 = \dot{x}_0 = 0$.

The initial values of $r, \theta, \dot{r}, \dot{\theta}$ will be denoted by $r_i, \theta_i, \dot{r}_i, \dot{\theta}_i$. Thus, by setting $t = 0$ in Equations 15 and 23

$$r_i = r_0 + \mu \left[-\frac{r_0}{6(\lambda+1)^2} - \frac{2B_1}{3(\lambda+1)} + k_1 + k_2 + k_3 + A_1 \right]. \quad (34)$$

By assuming $\theta_i = 0$ as usual, we have

$$B_2 = -2A_2 \quad (35)$$

from Equations 16 and 24. We have assumed $\dot{x}_0 = 0$ which is equivalent to $\dot{r}_i = 0$; hence

$$A_2 = B_2 = 0. \quad (36)$$

Finally, it is easy to obtain

$$\dot{\theta}_i = \lambda + \frac{\mu}{r_0} [l_1 \lambda + 2l_2 \lambda + 3l_3 \lambda - 2A_1(\lambda+1) + B_1]. \quad (37)$$

The initial values of r and $\dot{\theta}$ for the true periodic orbit will be denoted by $r_{i,p}$ and $\dot{\theta}_{i,p}$. They are given by

$$r_{i,p} = r_0 + \mu \left[-\frac{r_0}{6(\lambda+1)^2} + k_1 + k_2 + k_3 \right], \quad (38)$$

$$\dot{\theta}_{i,p} = \lambda + \frac{\mu\lambda}{r_0} (l_1 + 2l_2 + 3l_3). \quad (39)$$

From

$$\Delta r_i = r_i - r_{i,p}, \quad (40)$$

$$\Delta \dot{\theta}_i = \dot{\theta}_i - \dot{\theta}_{i,p}, \quad (41)$$

we derive

$$A_1 = -3 \frac{\Delta r_i}{\mu} - \frac{2r_0}{(\lambda+1)\mu} \frac{\Delta \theta_i}{\mu}, \quad (42)$$

$$B_1 = -6(\lambda+1) \frac{\Delta r_i}{\mu} - \frac{3r_0}{\mu} \frac{\Delta \theta_i}{\mu}. \quad (43)$$

We can now examine the behavior of orbits close to the periodic ones. The explicit expression for θ is, from Equations 16 and 24,

$$\theta = \left(\lambda + \frac{\mu}{r_0} B_1 \right) t + \frac{\mu}{r_0} \left[\sum_{i=1}^3 l_i \sin i\lambda t - 2A_1 \sin (\lambda+1) t \right]. \quad (44)$$

Let us define time as $t = t_n$ when $\theta = 2\pi n$, and $t = t_{n-1}$ when $\theta = 2\pi (n-1)$, where n is an integer. Therefore P_n , defined by

$$P_n = t_n - t_{n-1}, \quad (45)$$

is the period of the n^{th} cycle of a nearly periodic orbit. It follows from Equation 44 that

$$P_n = \frac{2\pi r_0}{\lambda r_0 + \mu B_1} - \frac{\mu}{\lambda r_0 + \mu B_1} \left[\sum_{i=1}^3 2l_i \sin \frac{i\lambda P_n}{2} \cos \frac{i\lambda (t_n + t_{n-1})}{2} - 4A_1 \sin \frac{(\lambda+1) P_n}{2} \cos \frac{(\lambda+1) (t_n + t_{n-1})}{2} \right], \quad (46)$$

from which we note immediately that the mean period of the nearly periodic orbit is

$$\langle P_n \rangle = \frac{2\pi r_0}{\lambda r_0 + \mu B_1}. \quad (47)$$

So the first term on the right-hand side of Equation 46 is associated with the mean period, but the rest denotes small oscillations of the P_n value around its mean value for various values of n .

Since the amplitudes of various oscillating terms in Equation 46 are small, λP_n may be set equal to 2π in the argument of the sine function. All terms involving l_i vanish as a result of this approximation and Equation 46 can be reduced to:

$$P_n = \langle P_n \rangle + \frac{4\mu A_1}{\lambda r_0 + \mu B_1} \sin \frac{\pi}{\lambda} \cos \left[(\lambda+1) t_n - \frac{\pi}{\lambda} \right], \quad (48)$$

where A_1 and B_1 are given by Equations 42 and 43, respectively, and t_n may be considered to be

$$t_n = n \langle P_n \rangle. \quad (49)$$

Since there is a periodic orbit for each value of λ in the range of interest, we can compare a nearly periodic orbit with any periodic orbit in the former's neighborhood. In our calculation we have fixed a single value of λ for both the periodic and nearly periodic orbit in order to derive A_1 and B_1 from Equations 42 and 43. Obviously a slightly different choice of the value of λ will give different values to these two constants for the same nearly periodic orbit.

As a simple example the two orbits may be compared by starting with the same r_i , i.e., $\Delta r_i = 0$. Then A_1 and B_1 are functions of $\Delta \dot{\theta}_i$ alone and $\Delta \dot{\theta}_i$ is related to the difference $\Delta \dot{y}_0$, between the initial value, \dot{y}_0 , of the nearly periodic orbit and that of the exactly periodic orbit. If, furthermore,

$$n \ll \frac{\lambda}{20\Delta \dot{\theta}_i}, \quad (50)$$

Equation 48 reduces to

$$P_n = \frac{2\pi}{\lambda - 3\Delta \dot{\theta}_i} - \frac{8\Delta \dot{\theta}_i}{\lambda(\lambda + 1)} \sin \frac{\pi}{\lambda} \cos \frac{(2n-1)\pi}{\lambda}. \quad (51)$$

From Equation 51 values of P_n have been computed up to $n = 28$ for two cases, with λ given by Equation 32. The results are given in the second and fourth columns of Table 3. In both cases $\lambda/\Delta \dot{\theta}_i$ is of the order of several thousand. Therefore, the use of Equation 51 is justified. For comparison P_n has also been computed by direct integration. This was done by computing successive times, t_n , when the orbit crossed the positive axis. A four point interpolation has been used for obtaining t_n from the integrated tables. In this way, from the initial conditions corresponding respectively to two values of \dot{y}_0 in Table 3, t_n , and consequently P_n , have been computed and the latter has been tabulated in the third and fifth

Table 3

Time Intervals for Successive Cycles of Nearly Periodic Orbits ($\mu = 0.012149$, $\lambda = 26.396884$).*

n	Time for $\dot{y}_0 = 2.8920000$		Time for $\dot{y}_0 = 2.8930000$	
	Equation 51*	Direct Integration	Equation 51*	Direct Integration
1	0.2378562	0.2378561	0.2380910	0.2380909
2	0.2378557	0.2378556	0.2380912	0.2380911
3	0.2378547	0.2378546	0.2380915	0.2380915
4	0.2378534	0.2378533	0.2380920	0.2380920
5	0.2378517	0.2378516	0.2380927	0.2380926
6	0.2378497	0.2378497	0.2380934	0.2380933
7	0.2378477	0.2378476	0.2380941	0.2380941
8	0.2378456	0.2378456	0.2380949	0.2380948
9	0.2378437	0.2378435	0.2380956	0.2380956
10	0.2378419	0.2378420	0.2380963	0.2380962
11	0.2378405	0.2378404	0.2380968	0.2380967
12	0.2378394	0.2378394	0.2380972	0.2380971
13	0.2378389	0.2378388	0.2380974	0.2380973
14	0.2378388	0.2378388	0.2380974	0.2380974
15	0.2378392	0.2378391	0.2380973	0.2380973
16	0.2378400	0.2378399	0.2380970	0.2380968
17	0.2378413	0.2378411	0.2380965	0.2380965
18	0.2378429	0.2378428	0.2380959	0.2380958
19	0.2378448	0.2378447	0.2380952	0.2380951
20	0.2378469	0.2378467	0.2380944	0.2380944
21	0.2378489	0.2378488	0.2380937	0.2380937
22	0.2378509	0.2378506	0.2380929	0.2380928
23	0.2378527	0.2378528	0.2380923	0.2380923
24	0.2378542	0.2378540	0.2380917	0.2380916
25	0.2378554	0.2378553	0.2380913	0.2380912
26	0.2378560	0.2378558	0.2380910	0.2380908
27	0.2378562	0.2378562	0.2380910	0.2380911
28	0.2378559	0.2378559	0.2380911	0.2380910

* In computing $\Delta \dot{\theta}_i$ the initial condition given by Equation 33 is considered to correspond to the exact periodic orbit used as the reference for comparison.

columns of Table 3. The agreement between the second and third columns as well as that between the fourth and fifth columns may be regarded as satisfactory.

SOME REMARKS ON RELATED PROBLEMS

Although the present calculation was performed in order to understand the general effect of the moon on the motion of the earth's satellites, it will serve equally well in understanding the general behavior of the motion of inner planets and asteroids as a result of the perturbation by the major planets, especially Jupiter. Indeed, the smallness of μ in such cases ($\mu = 9.539 \times 10^{-4}$ for the sun-Jupiter system) will make the result derived with the first order equations a good approximation to the problem. However, because of the relatively large sizes of the orbits of the inner planets, it may be necessary to include in the solution terms involving the third and perhaps higher powers of r_0 .

The coefficients k_n and l_n contain in their denominators a factor $(n\lambda)^2 - (\lambda + 1)^2$ where n is an integer. Thus if

$$n\lambda = \lambda + 1$$

the coefficients k_n and l_n diverge. Therefore, periodic orbits cannot be obtained in this way.

However, the Kirkwood gaps in the asteroid belt coincide with positions where asteroids, if present, would have periods commensurable with the period of Jupiter's orbit. Recently, this problem was investigated theoretically by Brouwer (Reference 8).

The presence of the Kirkwood gaps is obviously due to the perturbation by Jupiter. Since perturbation increases with μ , it follows that stronger gaps would be present in systems of two revolving bodies with increasing values of μ . Hence, we may immediately predict stronger gaps around the earth in the earth-moon system than those around the sun in the asteroid belt in the sun-Jupiter system. Such gaps around the earth can be computed easily from the condition of commensurability of the moon's period and the period of any satellite in such a zone. Of course, a result of this argument is that these zones should be avoided in launching satellites that are intended to stay in orbit for a long time. Being short, the period of the synchronous orbit is not near any strongly commensurable gap.

Another interesting consequence of the commensurable gaps around the earth may be found in the distribution of dust particles in the earth-moon system. It is not now known whether dust particles all move at random or partially revolve around the earth. If the latter should be the case, the presence of zones around the earth void of dust particles like the Kirkwood gaps in the asteroid belt around the sun would be inevitable. Consequently, their detection may be an interesting subject of investigation in the field of space research.

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